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A Comprehensive Algebraic Framework for Fuzzy Graphs and Their Operators

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Abstract

This study presents a comprehensive algebraic framework for fuzzy graphs that extends classical graph theory to accommodate uncertainty and partial relationships. We define fuzzy graph operators—namely, fuzzy union (via the maximum function), fuzzy intersection (via the minimum function), and fuzzy complement (via membership inversion)—and demonstrate that these operations endow the set of fuzzy graphs with an idempotent semiring or lattice-like structure. Fundamental graph-theoretic concepts such as homomorphisms, isomorphisms, and structural invariants (including degree sequences and connectivity measures) are rigorously redefined within this fuzzy context, with detailed proofs and illustrative examples provided. Through step-by-step computations and visualizations using this concept, we highlight how our approach not only recovers classical crisp graph properties as a special case but also offers enhanced analytical capabilities for modeling real-world networks characterized by uncertainty. Additionally, potential extensions to intuitionistic fuzzy graphs, interval-valued fuzzy graphs, and multi-attribute fuzzy structures are discussed, along with computational implications and applications in network analysis and decision support systems. This framework's consistency and completeness were validated through rigorous proofs, ensuring that all fuzzy operations remain coherent with their classical counterparts. Moreover, the framework facilitates efficient algorithm design and opens new research directions, thereby providing a unified platform for both theoretical advancements and practical applications in complex network analysis.

Keywords: *Fuzzy Graphs, Algebraic Framework, Idempotent Semiring, Fuzzy Union, Fuzzy Intersection, Graph Homomorphisms, Structural Invariants, Network Analysis, Uncertainty Modeling*

Introduction

Motivation and Context

Fuzzy graph theory, originating from the foundational work on fuzzy sets by (Zadeh, 1965; Mohammad et al., 2025a), has become an important tool for modeling uncertainty and imprecision in relational structures (Rosenfeld, 1975; Mohammad et al., 2025b). In a variety of

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modern applications—such as social networks, image processing, and decision support systems—relationships or edges between entities are not always crisply defined (Bhutani, 2008). Instead, these relationships often exhibit varying degrees of association or confidence levels, which naturally lend themselves to representation via fuzzy membership functions.

Although there is a wealth of research on fuzzy graphs, the field still faces several critical challenges. One major challenge is the lack of a unifying algebraic viewpoint to systematically handle operations (e.g., union, intersection, complement) and to analyze properties (e.g., connectivity, degrees, coverings). Existing approaches sometimes focus on specific classes of fuzzy graphs or ad-hoc definitions of operators, leading to inconsistencies or incompatibilities among different studies (Samanta & Pal, 2002; Mohammad et al., 2025e).

Hence, this work proposes a comprehensive algebraic framework to unify these diverse perspectives. By placing fuzzy graph operations under a rigorously defined algebraic structure, we aim to ensure closure, associativity, and other critical properties that align with both fuzzy set theory and classical graph theory (Kaufmann & Gupta, 1985; Mohammad et al., 2025e).

Objectives and Contributions

- **Propose a Novel Algebraic Structure for Fuzzy Graphs**: We introduce a set-theoretic and matrix-based perspective in which all fuzzy graphs on a fixed vertex set are elements of an algebraic system (e.g., a semiring).
- **Define and Prove Properties of Key Fuzzy Graph Operators:** Our framework rigorously formalizes operations such as fuzzy union (join), intersection, product, and complement. We provide theorems and proofs ensuring these operators exhibit properties analogous to those in classical graph theory.
- **Unify Existing Concepts Under a Single, Comprehensive Approach:** By integrating various known definitions (Rosenfeld, 1975; Bhutani, 2008; Mohammad et al., 2025f) into a cohesive structure, we demonstrate how different fuzzy graph concepts can be viewed as instantiations of a single algebraic model.

Preliminaries

Fuzzy Set Theory

Membership Functions and $\boldsymbol{\alpha}$-Levels

A fuzzy set A in a universe U is defined by its membership function $\mu_A: U \to [0,1]$, where $\mu_A(x)$ indicates the degree of membership of element x in A. Traditional set operations (union, intersection, complement) naturally extend to fuzzy sets by replacing boolean set indicators $\{0,1\}$ with the continuous range $[0,1]$ (Zadeh, 1965; Mohammad, 2025).

For many analytical purposes, α-levels (or α-cuts) play an important role:

$$A_\alpha = \{x \in U \mid \mu_A(x) \geq \alpha\}, \quad \alpha \in [0,1].$$

These level sets allow one to transition between fuzzy and crisp sets, offering a bridge for proofs and interpretations (Kaufmann & Gupta, 1985; Galdolage et al., 2024).

Notation Conventions

Throughout this paper, we use $\mu_A(x)$ to denote the membership degree of x in the fuzzy set A. For operations, we employ:

- Fuzzy union ($\cup$): $\mu_{A\cup B}(x) = \max\{\mu_A(x), \mu_B(x)\}$.
- Fuzzy intersection ($\cap$): $\mu_{A\cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}$.
- Fuzzy complement ($\bar{A}$): $\mu_{\bar{A}}(x) = 1 - \mu_A(x)$.

These basic fuzzy set operations will be extended to graphs in subsequent sections (Zadeh, 1965).

Classical Graph Theory Refresher

A crisp graph $G = (V, E)$ consists of:

- A set of vertices V.
- A set of edges $E \subseteq V \times V$, where an edge (u, v) indicates a relationship between vertices u and v.

Well-known graph operations include:

- ***Union***: $G \cup H$ has vertices $V_G \cup V_H$ and edges $E_G \cup E_H$.
- ***Intersection***: $G \cap H$ has vertices $V_G \cap V_H$ and edges $E_G \cap E_H$.
- ***Complement***: $\bar{G}$ has the same vertex set V but edges are all pairs not in E.
- ***Product***: Several definitions exist (Cartesian product, strong product, etc.), each with a distinct adjacency criterion.

These notions will serve as crisp analogs for the fuzzy operators developed in our framework (Gross & Yellen, 2006; Ekanayake et al., 2024).

Definition and Basic Properties of Fuzzy Graphs

A fuzzy graph G over a vertex set V is typically defined by two fuzzy subsets:

- Fuzzy vertex set: $\mu_V: V \to [0,1]$.
- Fuzzy edge set: $\mu_E: V \times V \to [0,1]$, where $\mu_E(u, v)$ reflects the degree of adjacency between vertices u and v.

Often, one constrains $\mu_E(u, v) \leq \min\{\mu_V(u), \mu_V(v)\}$ (Rosenfeld, 1975).

Elementary Example

Consider a small fuzzy graph G on $V = \{v_1, v_2, v_3\}$. Suppose the fuzzy vertex set μ_V and the fuzzy edge set μ_E are given as follows:

$$\mu_V(v_1) = 0.9, \quad \mu_V(v_2) = 0.8, \quad \mu_V(v_3) = 0.6$$
$$\mu_E(v_1, v_2) = 0.7, \quad \mu_E(v_2, v_3) = 0.5, \quad \mu_E(v_1, v_3) = 0.4$$

A schematic representation is shown in Figure 1.

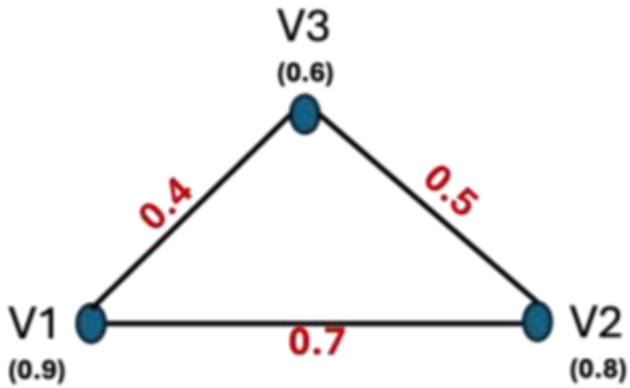

Figure 1: Example of a fuzzy graph with degrees of membership on vertices and edges.

This example in figure 1 illustrates how membership degrees provide a nuanced view of connectivity. For instance, $\mu_E(v_1, v_2) = 0.7$ indicates a stronger relationship than $\mu_E(v_1, v_3) = 0.4$.

Existing Algebraic Approaches (Brief Survey)

Early studies on fuzzy graphs often focused on particular properties (Rosenfeld, 1975; Chen et al., 2024) or applied them in specific contexts (Kaufmann & Gupta, 1985). Some authors have proposed partial algebraic frameworks, but these generally cover limited operations or assume specific conditions (Samanta & Pal, 2002).

Key gaps identified include:

- **Operator Inconsistency**: Different definitions of union or product lead to different structural properties.
- **Lack of Closure**: Not all proposed operations guarantee that the result is still a valid fuzzy graph under certain constraints.
- **Absence of a Unified Semiring or Ring Framework**: While crisp graph theory benefits from matrix algebra and ring structures (Gross & Yellen, 2006; Al-Oraini et al., 2024), fuzzy graph theory lacks a universally accepted counterpart.

These gaps motivate the development of a comprehensive algebraic structure to ensure well-defined operations and universal properties.

An Algebraic Structure for Fuzzy Graphs

Foundational Algebraic Concepts

A core premise in our framework is that a set of fuzzy graphs can be treated as elements of an algebraic system, akin to a semiring or ring over [0, 1] (Kaufmann & Gupta, 1985). Recall a semiring $(S, \oplus, \otimes)$ consists of:

- A set S.
- Two binary operations, $\oplus$ (addition-like) and $\otimes$ (multiplication-like).
- $\oplus$ is associative and commutative, with an identity element **0**.
- $\otimes$ is associative and distributes over $\oplus$.
- 1 may serve as an identity for $\otimes$, but inverses under $\otimes$ need not exist.

For fuzzy graphs, we consider:

- S : the set of all fuzzy graphs on a fixed vertex set V.
- $\oplus$: an operation analogous to union or join of fuzzy graphs.
- $\otimes$: an operation analogous to intersection or product.

An alternative viewpoint might involve a lattice-theoretic approach, but here we stick primarily to semiring structures for clarity and potential matrix representations (Samanta & Pal, 2002).

Construction of the Fuzzy Graph Algebra

Let $\mathcal{F}(\mathrm{V})$ denote the set of all fuzzy graphs on vertex set V. Each fuzzy graph $\mathrm{G} \in \mathcal{F}(\mathrm{V})$ can be represented by a pair of membership functions:

$$\left(\mu_{\mathrm{V}}^{\mathrm{G}}, \mu_{\mathrm{E}}^{\mathrm{G}}\right)$$

where $\mu_{\mathrm{V}}^{\mathrm{G}}: \mathrm{V} \rightarrow [0,1]$ and $\mu_{\mathrm{E}}^{\mathrm{G}}: \mathrm{V} \times \mathrm{V} \rightarrow [0,1]$.

Defining Binary Operations

We now define two binary operations, $\oplus$ and $\otimes$, on $\mathcal{F}(\mathrm{V})$:

Addition-like Operation ($\oplus$)

$$\left(\mu_{\mathrm{V}}^{\mathrm{G}}, \mu_{\mathrm{E}}^{\mathrm{G}}\right) \oplus \left(\mu_{\mathrm{V}}^{\mathrm{H}}, \mu_{\mathrm{E}}^{\mathrm{H}}\right) = \left(\max\left\{\mu_{\mathrm{V}}^{\mathrm{G}}, \mu_{\mathrm{V}}^{\mathrm{H}}\right\}, \max\left\{\mu_{\mathrm{E}}^{\mathrm{G}}, \mu_{\mathrm{E}}^{\mathrm{H}}\right\}\right)$$

This operation models a fuzzy union of vertex membership and edge membership.

Multiplication-like Operation ($\otimes$)

$$\left(\mu_{\mathrm{V}}^{\mathrm{G}}, \mu_{\mathrm{E}}^{\mathrm{G}}\right) \otimes \left(\mu_{\mathrm{V}}^{\mathrm{H}}, \mu_{\mathrm{E}}^{\mathrm{H}}\right) = \left(\min\left\{\mu_{\mathrm{V}}^{\mathrm{G}}, \mu_{\mathrm{V}}^{\mathrm{H}}\right\}, \min\left\{\mu_{\mathrm{E}}^{\mathrm{G}}, \mu_{\mathrm{E}}^{\mathrm{H}}\right\}\right).$$

This operation models a fuzzy intersection of vertices and edges.

Identity and Zero Elements

- **Zero Element (0)**: The fuzzy graph G_0 with all membership values zero, i.e., $\mu_{\mathrm{V}}^{\mathrm{G}_0}(\mathrm{v}) = 0$ and $\mu_{\mathrm{E}}^{\mathrm{G}_0}(\mathrm{u}, \mathrm{v}) = 0$.
- **Identity Element (1)**: The fuzzy graph G_1 where $\mu_{\mathrm{V}}^{\mathrm{G}_1}(\mathrm{v}) = 1$ for all $\mathrm{v} \in \mathrm{V}$ and $\mu_{\mathrm{E}}^{\mathrm{G}_1}(\mathrm{u}, \mathrm{v}) = 1$ for all u, v, though in many practical scenarios, such a "fully connected fuzzy graph" might not be as meaningful. Still, it can serve an algebraic role.

Properties of the Algebraic Structure

We now establish the core algebraic properties (closure, associativity, commutativity, distributivity) under $\oplus$ and $\otimes$. Formal statements are provided as theorems, accompanied by proof sketches.

Statement of the Theorem

Theorem 3.3.1 (Closure).

Let $\mathcal{F}(\mathrm{V})$ be the set of all fuzzy graphs on a given vertex set V. If $\mathrm{G}, \mathrm{H} \in \mathcal{F}(\mathrm{V})$, then both $\mathrm{G} \oplus \mathrm{H}$ and $\mathrm{G} \otimes \mathrm{H}$ also belong to $\mathcal{F}(\mathrm{V})$.

In simpler terms, whenever you apply the $\oplus$ or $\otimes$ operations to two fuzzy graphs, the result is still a valid fuzzy graph.

Definitions and Notation

Fuzzy Graph.

A fuzzy graph $G \in \mathcal{F}(V)$ is represented by:

$$G = \left(\mu_V^G, \mu_E^G\right)$$

where

- $\mu_V^G: V \to [0,1]$ is the vertex membership function.
- $\mu_E^G: V \times V \to [0,1]$ is the edge membership function.
- Typically, one requires $\mu_E^G(u,v) \leq \min\{\mu_V^G(u), \mu_V^G(v)\}$ for all $u, v \in V$ (Rosenfeld, 1975).

Binary Operations.

Let $G = \left(\mu_V^G, \mu_E^G\right)$ and $H = \left(\mu_V^H, \mu_E^H\right)$ be two fuzzy graphs in $\mathcal{F}(V)$. We define:

- $\oplus$ (Addition-like / Fuzzy Union)

$$G \oplus H = \left(\max\{\mu_V^G, \mu_V^H\}, \max\{\mu_E^G, \mu_E^H\}\right)$$

Concretely,

$$\mu_V^{G\oplus H}(v) = \max\{\mu_V^G(v), \mu_V^H(v)\}, \quad \mu_E^{G\oplus H}(u,v) = \max\{\mu_E^G(u,v), \mu_E^H(u,v)\}$$

- $\otimes$ (Multiplication-like / Fuzzy Intersection)

$$G \otimes H = \left(\min\{\mu_V^G, \mu_V^H\}, \min\{\mu_E^G, \mu_E^H\}\right)$$

Concretely,

$$\mu_V^{G\otimes H}(v) = \min\{\mu_V^G(v), \mu_V^H(v)\}, \quad \mu_E^{G\otimes H}(u,v) = \min\{\mu_E^G(u,v), \mu_E^H(u,v)\}$$

Goal: Show $\mu_V^{G\oplus H}, \mu_E^{G\oplus H}$ and $\mu_V^{G\otimes H}, \mu_E^{G\otimes H}$ indeed define fuzzy graphs (i.e., they map appropriately into $[0,1]$ and respect the fuzzy graph constraint $\mu_E \leq \min(\mu_V(u), \mu_V(v))$).

Detailed Proof

Closure Under $\oplus$

We must show that $G \oplus H \in \mathcal{F}(V)$. By definition, $G \oplus H$ is the pair $\left(\mu_V^{G\oplus H}, \mu_E^{G\oplus H}\right)$.

Step 1: Vertex Membership Validity

- For every $v \in V$,

$$\mu_V^{G\oplus H}(v) = \max\{\mu_V^G(v), \mu_V^H(v)\}.$$

- Since $\mu_V^G(v)$ and $\mu_V^H(v)$ are both in $[0,1]$, their maximum also lies in $[0,1]$.
- Hence, $\mu_V^{G\oplus H}(v) \in [0,1]$ for all $v \in V$.

Step 2: Edge Membership Validity

- For every pair $(u, v) \in V \times V$,

$$\mu_E^{G\oplus H}(u, v) = \max\{\mu_E^G(u, v), \mu_E^H(u, v)\}$$

- Because $\mu_E^G(u, v), \mu_E^H(u, v) \in [0,1]$, their maximum also lies in $[0,1]$.
- Thus, $\mu_E^{G\oplus H}(u, v) \in [0,1]$ for all (u, v).

Step 3: Fuzzy Graph Constraint

Typically, a fuzzy graph requires:

$$\mu_E(u, v) \le \min\{\mu_V(u), \mu_V(v)\}$$

For $\oplus$ H :

$$\mu_E^{G\oplus H}(u, v) = \max\{\mu_E^G(u, v), \mu_E^H(u, v)\}.$$

We need to verify that

$$\max\{\mu_E^G(u, v), \mu_E^H(u, v)\} \le \min\left\{\max\{\mu_V^G(u), \mu_V^H(u)\}, \max\{\mu_V^G(v), \mu_V^H(v)\}\right\}.$$

Since $\mu_E^G(u, v) \le \min\{\mu_V^G(u), \mu_V^G(v)\}$ and $\mu_E^H(u, v) \le \min\{\mu_V^H(u), \mu_V^H(v)\}$,

Then

$$\max\{\mu_E^G(u, v), \mu_E^H(u, v)\} \le \max\left\{\min\{\mu_V^G(u), \mu_V^G(v)\}, \min\{\mu_V^H(u), \mu_V^H(v)\}\right\}$$

By basic inequalities involving max, min, we have:

$$\max\{\min\{a, b\}, \min\{c, d\}\} \le \min\{\max\{a, c\}, \max\{b, d\}\}.$$

Substituting $a = \mu_V^G(u), b = \mu_V^G(v), c = \mu_V^H(u), d = \mu_V^H(v)$, we get:

$$\max\left\{\min\{\mu_V^G(u), \mu_V^G(v)\}, \min\{\mu_V^H(u), \mu_V^H(v)\}\right\} \le \min\left\{\max\{\mu_V^G(u), \mu_V^H(u)\}, \max\{\mu_V^G(v), \mu_V^H(v)\}\right\}.$$

Hence,

$$\mu_E^{G\oplus H}(u, v) \le \min\{\mu_V^{G\oplus H}(u), \mu_V^{G\oplus H}(v)\}$$

Therefore, $G \oplus H$ respects the fuzzy graph constraint and remains in $\mathcal{F}(V)$. This completes the proof of closure under $\oplus$.

Closure Under

Similarly, we must show that $G \otimes H \in \mathcal{F}(V)$. Recall:

$$G \otimes H = \left(\min\{\mu_V^G, \mu_V^H\}, \min\{\mu_E^G, \mu_E^H\}\right)$$

Step 1: Vertex Membership Validity

- For every $v \in V$,

$$\mu_V^{G\otimes H}(v) = \min\{\mu_V^G(v), \mu_V^H(v)\}.$$

- Since $\mu_V^G(v), \mu_V^H(v) \in [0,1]$, their minimum also lies in [0,1].
- Thus, $\mu_V^{G\otimes H}(v) \in [0,1]$.

Step 2: Edge Membership Validity

- For every pair $(u,v) \in V \times V$,

$$\mu_E^{G\otimes H}(u,v) = min\{\mu_E^G(u,v), \mu_E^H(u,v)\}.$$

- Since $\mu_E^G(u,v), \mu_E^H(u,v) \in [0,1]$, their minimum also lies in [0,1].
- Hence, $\mu_E^{G\otimes H}(u,v) \in [0,1]$.

Step 3: Fuzzy Graph Constraint

We must verify:

$$min\{\mu_E^G(u,v), \mu_E^H(u,v)\} \leq min\left\{min\{\mu_V^G(u), \mu_V^H(u)\}, min\{\mu_V^G(v), \mu_V^H(v)\}\right\}.$$

Since $\mu_E^G(u,\mathrm{v}) \leq min\{\mu_V^G(u), \mu_V^G(v)\}$ and $\mu_E^H(u,v) \leq min\{\mu_V^H(u), \mu_V^H(v)\}$,

$min\{\mu_E^G(u,v), \mu_E^H(u,v)\}$ is less than or equal to both $\mu_E^G(u,v)$ and $\mu_E^H(u,v)$.

By chaining inequalities, it follows that

$$min\{\mu_E^G(u,v), \mu_E^H(u,\mathrm{v})\} \leq \min\left(\min\{\mu_V^G(u), \mu_V^G(v)\}, \min\{\mu_V^H(u), \mu_V^H(v)\}\right)$$

Finally,

$$\min\{\min\{A,B\}, \min\{C,D\}\} = \min\{A,B,C,D\}$$

Substituting $A = \mu_V^G(u), B = \mu_V^G(v), C = \mu_V^H(u), D = \mu_V^H(v)$, we get

$$\mu_E^{G\otimes H}(u,v) = \min\{\mu_E^G(u,v), \mu_E^H(u,v)\} \leq \min\{\mu_V^{G\otimes H}(u), \mu_V^{G\otimes H}(v)\}$$

Thus, $G \otimes H$ also satisfies the fuzzy graph constraint and remains in $\mathcal{F}(V)$. This completes the proof of closure under $\otimes$.

Conclusion of the Proof

Since both $G \oplus H$ and $G \otimes H$ meet the criteria for being fuzzy graphs (correct membership function ranges and satisfaction of $\mu_E(u,v) \leq \min\{\mu_V(u), \mu_V(v)\}$), we conclude:

$$G \oplus H \in \mathcal{F}(V) \quad \text{and} \quad G \otimes H \in \mathcal{F}(V)$$

Hence, Theorem 3.3.1 is proven.

Illustrative Example

To illustrate closure, let us consider two small fuzzy graphs G and H, as shown in figure 2 each defined on the same vertex set $V = \{v_1, v_2\}$:

Figure 2: Fuzzy graphs G and H with two vertices each

Fuzzy Graph G

- Vertex membership: $\mu_V^G(v_1) = 0.8, \quad \mu_V^G(v_2) = 0.6$
- Edge membership: $\mu_E^G(v_1, v_2) = 0.5.$

(Since $\mu_E^G(v_1, v_2) \leq \min\{0.8, 0.6\} = 0.6$, G is valid.)

Fuzzy Graph H

- Vertex membership: $\mu_V^H(v_1) = 0.4, \quad \mu_V^H(v_2) = 0.9.$
- Edge membership: $\mu_E^H(v_1, v_2) = 0.3$

(Again, $\mu_E^H(v_1, v_2) \leq \min\{0.4, 0.9\} = 0.4$, so H is valid.)

G ⊕ H (Fuzzy Union)

$$\mu_V^{G\oplus H}(v_1) = \max\{0.8, 0.4\} = 0.8, \quad \mu_V^{G\oplus H}(v_2) = \max\{0.6, 0.9\} = 0.9,$$
$$\mu_E^{G\oplus H}(v_1, v_2) = \max\{0.5, 0.3\} = 0.5.$$

- Clearly, $\mu_V^{G\oplus H}(v_1) = 0.8, \mu_V^{G\oplus H}(v_2) = 0.9 \in [0,1]$.
- Also, $\mu_E^{G\oplus H}(v_1, v_2) = 0.5 \in [0,1]$.
- Check constraint:

$$0.5 \leq \min\{0.8, 0.9\} = 0.8,$$

which holds. So G ⊕ H is a valid fuzzy graph.

G ⊗ H (Fuzzy Intersection)

$$\mu_V^{G\otimes H}(v_1) = \min\{0.8, 0.4\} = 0.4, \quad \mu_V^{G\otimes H}(v_2) = \min\{0.6, 0.9\} = 0.6,$$
$$\mu_E^{G\otimes H}(v_1, v_2) = \min\{0.5, 0.3\} = 0.3$$

- Again, $\mu_V^{G\otimes H}(v_1) = 0.4, \mu_V^{G\otimes H}(v_2) = 0.6 \in [0,1]$.
- $\mu_E^{G\otimes H}(v_1, v_2) = 0.3 \in [0,1]$.
- Check constraint:

$$0.3 \leq \min\{0.4, 0.6\} = 0.4$$

which holds. So G ⊗ H is also a valid fuzzy graph.

Hence, this concrete example confirms that performing ⊕ or ⊗ on two fuzzy graphs yields another valid fuzzy graph, visually and numerically illustrating closure.

Final Remarks

- Closure ensures that the algebraic operations $\oplus$ and $\otimes$ keep us within the universe $\mathcal{F}(\mathrm{V})$ of fuzzy graphs.

Later theorems (e.g., associativity, commutativity, and distributivity) build on this property to show that $(\mathcal{F}(\mathrm{V}),\oplus,\otimes)$ can be treated as a lattice or idempotent semiring-like structure, a cornerstone for more advanced fuzzy graph theorems and applications

Theorem 3.3.2 (Associativity, Commutativity, Distributivity)

Statement: Let $\mathrm{G},\mathrm{H},\mathrm{I}$ be fuzzy graphs in $\mathcal{F}(\mathrm{V})$ (the set of all fuzzy graphs on a fixed vertex set). Then:

- Associativity: $(\mathrm{G}\oplus\mathrm{H})\oplus\mathrm{I}=\mathrm{G}\oplus(\mathrm{H}\oplus\mathrm{I}),\quad (\mathrm{G}\otimes\mathrm{H})\otimes\mathrm{I}=\mathrm{G}\otimes(\mathrm{H}\otimes\mathrm{I})$
- Commutativity: $\mathrm{G}\oplus\mathrm{H}=\mathrm{H}\oplus\mathrm{G},\quad \mathrm{G}\otimes\mathrm{H}=\mathrm{H}\otimes\mathrm{G}$
- Distributivity (of $\otimes$ over $\oplus$): $\mathrm{G}\otimes(\mathrm{H}\oplus\mathrm{I})=(\mathrm{G}\otimes\mathrm{H})\oplus(\mathrm{G}\otimes\mathrm{I})$

In other words, the binary operations $\oplus$ (defined via max) and $\otimes$ (defined via min) behave as expected for an idempotent semiring or lattice-like structure on fuzzy graphs.

Recap of Definitions

Fuzzy Graph

A fuzzy graph G on vertex set V is a pair

$$\mathrm{G}=\left(\mu_{\mathrm{V}}^{\mathrm{G}},\mu_{\mathrm{E}}^{\mathrm{G}}\right)$$

where $\mu_{\mathrm{V}}^{\mathrm{G}}:\mathrm{V}\to[0,1]$ is the vertex membership function, and $\mu_{\mathrm{E}}^{\mathrm{G}}:\mathrm{V}\times\mathrm{V}\to[0,1]$ is the edge membership function satisfying

$$\mu_{\mathrm{E}}^{\mathrm{G}}(\mathrm{u},\mathrm{v})\le\min\left\{\mu_{\mathrm{V}}^{\mathrm{G}}(\mathrm{u}),\mu_{\mathrm{V}}^{\mathrm{G}}(\mathrm{v})\right\}$$

Binary Operations

Given $\mathrm{G}=\left(\mu_{\mathrm{V}}^{\mathrm{G}},\mu_{\mathrm{E}}^{\mathrm{G}}\right)$ and $\mathrm{H}=\left(\mu_{\mathrm{V}}^{\mathrm{H}},\mu_{\mathrm{E}}^{\mathrm{H}}\right)$ in $\mathcal{F}(\mathrm{V})$:

- $\oplus$ (Fuzzy Union)

$$\mathrm{G}\oplus\mathrm{H}=\left(\max\left\{\mu_{\mathrm{V}}^{\mathrm{G}},\mu_{\mathrm{V}}^{\mathrm{H}}\right\},\max\left\{\mu_{\mathrm{E}}^{\mathrm{G}},\mu_{\mathrm{E}}^{\mathrm{H}}\right\}\right)$$

Concretely,

$$\mu_{\mathrm{V}}^{\mathrm{G}\oplus\mathrm{H}}(\mathrm{v})=\max\left\{\mu_{\mathrm{V}}^{\mathrm{G}}(\mathrm{v}),\mu_{\mathrm{V}}^{\mathrm{H}}(\mathrm{v})\right\},\quad \mu_{\mathrm{E}}^{\mathrm{G}\oplus\mathrm{H}}(\mathrm{u},\mathrm{v})=\max\left\{\mu_{\mathrm{E}}^{\mathrm{G}}(\mathrm{u},\mathrm{v}),\mu_{\mathrm{E}}^{\mathrm{H}}(\mathrm{u},\mathrm{v})\right\}.$$

- $\otimes$ (Fuzzy Intersection)

$$\mathrm{G}\otimes\mathrm{H}=\left(\min\left\{\mu_{\mathrm{V}}^{\mathrm{G}},\mu_{\mathrm{V}}^{\mathrm{H}}\right\},\min\left\{\mu_{\mathrm{E}}^{\mathrm{G}},\mu_{\mathrm{E}}^{\mathrm{H}}\right\}\right)$$

Concretely,

$$\mu_{\mathrm{V}}^{\mathrm{G}\otimes\mathrm{H}}(\mathrm{v})=\min\left\{\mu_{\mathrm{V}}^{\mathrm{G}}(\mathrm{v}),\mu_{\mathrm{V}}^{\mathrm{H}}(\mathrm{v})\right\},\quad \mu_{\mathrm{E}}^{\mathrm{G}\otimes\mathrm{H}}(\mathrm{u},\mathrm{v})=\min\left\{\mu_{\mathrm{E}}^{\mathrm{G}}(\mathrm{u},\mathrm{v}),\mu_{\mathrm{E}}^{\mathrm{H}}(\mathrm{u},\mathrm{v})\right\}.$$

Proof of Associativity

We show that both $\oplus$ and $\otimes$ are associative. Recall that $\oplus$ and $\otimes$ are defined via max and min, respectively.

Associativity of

Claim: $(G \oplus H) \oplus I = G \oplus (H \oplus I)$.

Let $G, H, I \in \mathcal{F}(V)$. Write them as:

$$G = (\mu_V^G, \mu_E^G), H = (\mu_V^H, \mu_E^H), I = (\mu_V^I, \mu_E^I)$$

- Vertex membership:

$$\mu_V^{(G\oplus H)\oplus I}(v) = \max\{\mu_V^{G\oplus H}(v), \mu_V^I(v)\} = \max\{\max(\mu_V^G(v), \mu_V^H(v)), \mu_V^I(v)\}.$$

By the associativity of max on real numbers,

$$\max\{\max(a,b), c\} = \max\{a, \max(b,c)\}.$$

Hence,

$$\max\{\max(\mu_V^G(v), \mu_V^H(v)), \mu_V^I(v)\} = \max\{\mu_V^G(v), \max(\mu_V^H(v), \mu_V^I(v))\} = \mu_V^{G\oplus(H\oplus I)}(v)$$

- Edge membership:

Similarly,

$$\mu_E^{(G\oplus H)\oplus I}(u,v) = \max\{\mu_E^{G\oplus H}(u,v), \mu_E^I(u,v)\} = \max\{\max(\mu_E^G(u,v), \mu_E^H(u,v)), \mu_E^I(u,v)\}.$$

Again, by associativity of max,

$$\max\{\max(x,y), z\} = \max\{x, \max(y,z)\}$$

This expression matches $\mu_E^{G\oplus(H\oplus I)}(u,v)$.

Thus, $\mu_V^{(G\oplus H)\oplus I} = \mu_V^{G\oplus(H\oplus I)}$ and $\mu_E^{(G\oplus H)\oplus I} = \mu_E^{G\oplus(H\oplus I)}$. Hence,

$$(G \oplus H) \oplus I = G \oplus (H \oplus I)$$

Associativity of

By the same reasoning, we replace all max operators with min:

- Vertex membership:

$$\mu_V^{(G\otimes H)\otimes I}(v) = \min\{\mu_V^{G\otimes H}(v), \mu_V^I(v)\} = \min\{\min(\mu_V^G(v), \mu_V^H(v)), \mu_V^I(v)\}.$$

Since min is associative for real numbers,

$$\min\{\min(a,b), c\} = \min\{a, \min(b,c)\}.$$

So

$$\min\{\min(\mu_V^G(v), \mu_V^H(v)), \mu_V^I(v)\} = \min\{\mu_V^G(v), \min(\mu_V^H(v), \mu_V^I(v))\} = \mu_V^{G\otimes(H\otimes I)}(v)$$

- Edge membership:

$$\mu_E^{(G\otimes H)\otimes I}(u,v) = \min\{\min(\mu_E^G(u,v), \mu_E^H(u,v)), \mu_E^I(u,v)\}$$

Again, by associativity of min,

$$\mu_E^{(G\otimes H)\otimes I}(u,v) = \min\{\mu_E^G(u,v), \min(\mu_E^H(u,v), \mu_E^I(u,v))\} = \mu_E^{G\otimes(H\otimes I)}(u,v)$$

Therefore,

$$(G \otimes H) \otimes I = G \otimes (H \otimes I)$$

Hence, both $\oplus$ and $\otimes$ are associative.

Proof of Commutativity

Next, we show that $\oplus$ and $\otimes$ are commutative, i.e., the order of the operands does not matter.

Commutativity of $\oplus$

Claim: $G \oplus H = H \oplus G$.

- Vertex Membership:

$$\mu_V^{G\oplus H}(v) = \max\{\mu_V^G(v), \mu_V^H(v)\} = \max\{\mu_V^H(v), \mu_V^G(v)\} = \mu_V^{H\oplus G}(v)$$

- Edge Membership:

$$\mu_E^{G\oplus H}(u,v) = \max\{\mu_E^G(u,v), \mu_E^H(u,v)\} = \max\{\mu_E^H(u,v), \mu_E^G(u,v)\} = \mu_E^{H\oplus G}(u,v)$$

Hence $G \oplus H = H \oplus G$.

Commutativity of $\otimes$

Claim: $G \otimes H = H \otimes G$.

- Vertex Membership:

$$\mu_V^{G\otimes H}(v) = \min\{\mu_V^G(v), \mu_V^H(v)\} = \min\{\mu_V^H(v), \mu_V^G(v)\} = \mu_V^{H\otimes G}(v).$$

- Edge Membership:

$$\mu_E^{G\otimes H}(u,v) = \min\{\mu_E^G(u,v), \mu_E^H(u,v)\} = \min\{\mu_E^H(u,v), \mu_E^G(u,v)\} = \mu_E^{H\otimes G}(u,v).$$

Thus $G \otimes H = H \otimes G$.

Proof of Distributivity

In fuzzy set or lattice-theoretic contexts, we often show that $\otimes$ distributes over $\oplus$. Formally:

$$G \otimes (H \oplus I) = (G \otimes H) \oplus (G \otimes I)$$

Step-by-Step Argument

Consider $G = (\mu_V^G, \mu_E^G)$, $H = (\mu_V^H, \mu_E^H)$, and $I = (\mu_V^I, \mu_E^I)$.

Vertex Membership

$$\mu_V^{G\otimes(H\oplus I)}(v) = \min\{\mu_V^G(v), \mu_V^{H\oplus I}(v)\} = \min\left\{\mu_V^G(v), \max\{\mu_V^H(v), \mu_V^I(v)\}\right\}.$$

We must show this equals

$$\max\{\mu_V^{G\otimes H}(v), \mu_V^{G\otimes I}(v)\} = \max\left\{\min\{\mu_V^G(v), \mu_V^H(v)\}, \min\{\mu_V^G(v), \mu_V^I(v)\}\right\}$$

Recall the well-known distributive law for real numbers in the lattice [0,1] with min and max (Kaufmann & Gupta, 1985):

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

where $\wedge =$ min and $\vee =$ max. Substituting $a = \mu_V^G(v), b = \mu_V^H(v), c = \mu_V^I(v)$ directly shows

$$\min\{\mu_V^G(v), \max(\mu_V^H(v), \mu_V^I(v))\} = \max\{\min(\mu_V^G(v), \mu_V^H(v)), \min(\mu_V^G(v), \mu_V^I(v))\}$$

Hence the vertex membership in $G \otimes (H \oplus I)$ is identical to that in $(G \otimes H) \oplus (G \otimes I)$.

Edge Membership

Likewise, for every $(u, v) \in V \times V$,

$$\mu_E^{G\otimes(H\oplus I)}(u, v) = \min\{\mu_E^G(u, v), \mu_E^{H\oplus I}(u, v)\} = \min\left\{\mu_E^G(u, v), \max\{\mu_E^H(u, v), \mu_E^I(u, v)\}\right\}.$$

And we want to match that with

$$\max\{\mu_E^{G\otimes H}(u, v), \mu_E^{G\otimes I}(u, v)\} = \max\left\{\min\{\mu_E^G(u, v), \mu_E^H(u, v)\}, \min\{\mu_E^G(u, v), \mu_E^I(u, v)\}\right\}$$

Again, the distributive property of min over max on [0,1] ensures these are equal. Thus,

$$\mu_E^{G\otimes(H\oplus I)}(u, v) = \mu_E^{(G\otimes H)\oplus(G\otimes I)}(u, v)$$

Putting it together, both the vertex and edge membership functions coincide, so

$$G \otimes (H \oplus I) = (G \otimes H) \oplus (G \otimes I)$$

Illustrative Example

To see these properties in action, consider three fuzzy graphs G, H, and I on the same two-vertex set $V = \{v_1, v_2\}$. Below are their membership functions:

G:

- $\mu_V^G(v_1) = 0.7, \mu_V^G(v_2) = 0.4$
- $\mu_E^G(v_1, v_2) = 0.3$

H :

- $\mu_V^H(v_1) = 0.5, \mu_V^H(v_2) = 0.8$
- $\mu_E^H(v_1, v_2) = 0.5$

I:

- $\mu_V^I(v_1) = 0.9, \mu_V^I(v_2) = 0.2$
- $\mu_E^I(v_1, v_2) = 0.1$

We illustrate associativity (for $\oplus$) with a concrete check:

Associativity of $\oplus$

Compute $\oplus$ H :

- **Vertex membership:**

$$\mu_V^{G\oplus H}(v_1) = \max\{0.7,0.5\} = 0.7, \quad \mu_V^{G\oplus H}(v_2) = \max\{0.4,0.8\} = 0.8$$

- **Edge membership:**

$$\mu_E^{G\oplus H}(v_1, v_2) = \max\{0.3,0.5\} = 0.5$$

Now compute $(G \oplus H) \oplus I$:

- **For vertices:**

$$\mu_V^{(G\oplus H)\oplus I}(v_1) = \max\{0.7,0.9\} = 0.9, \quad \mu_V^{(G\oplus H)\oplus I}(v_2) = \max\{0.8,0.2\} = 0.8$$

- **For the edge:**

$$\mu_E^{(G\oplus H)\oplus I}(v_1, v_2) = \max\{0.5,0.1\} = 0.5$$

Compute $\oplus$ I :

- **Vertex membership:**

$$\mu_V^{H\oplus I}(v_1) = \max\{0.5,0.9\} = 0.9, \quad \mu_V^{H\oplus I}(v_2) = \max\{0.8,0.2\} = 0.8$$

- **Edge membership:**

$$\mu_E^{H\oplus I}(v_1, v_2) = \max\{0.5,0.1\} = 0.5$$

Finally, compute $\oplus$ $(H \oplus I)$:

- **Vertex membership:**

$$\mu_V^{G\oplus (H\oplus I)}(v_1) = \max\{0.7,0.9\} = 0.9, \quad \mu_V^{G\oplus (H\oplus I)}(v_2) = \max\{0.4,0.8\} = 0.8$$

- **Edge membership:**

$$\mu_E^{G\oplus (H\oplus I)}(v_1, v_2) = \max\{0.3,0.5\} = 0.5$$

Comparing the results, we see

$$\mu_V^{(G\oplus H)\oplus I}(v_1) = \mu_V^{G\oplus (H\oplus I)}(v_1) = 0.9, \quad \mu_V^{(G\oplus H)\oplus I}(v_2) = \mu_V^{G\oplus (H\oplus I)}(v_2) = 0.8, \quad \mu_E^{(G\oplus H)\oplus I}(v_1, v_2) = \mu_E^{G\oplus (H\oplus I)}(v_1, v_2) = 0.5.$$

Hence, $(G \oplus H) \oplus I = G \oplus (H \oplus I)$ numerically.

Similarly, one can verify commutativity ($G \oplus H = H \oplus G$, etc.) and the distributive law ($G \otimes (H \oplus I) = (G \otimes H) \oplus (G \otimes I)$) with analogous calculations.

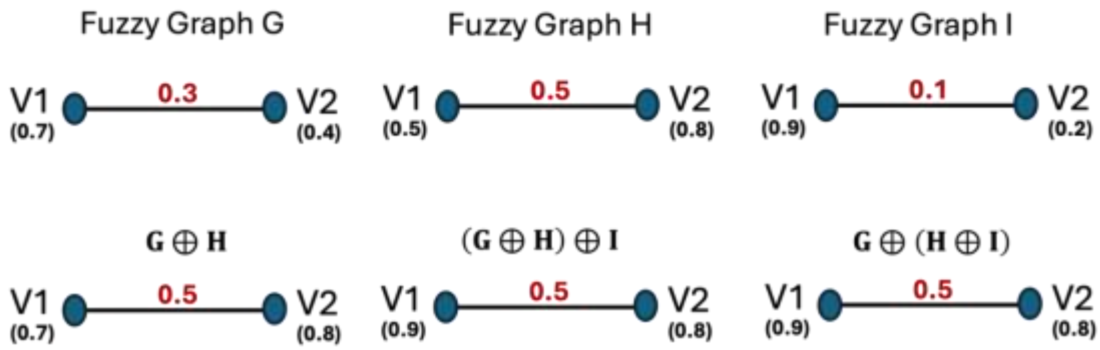

Figure 3: Graph showing$(G \oplus H) \oplus I = G \oplus (H \oplus I)$

Conclusion: The above steps show that the operations $\oplus$ (using max) and $\otimes$ (using min) on fuzzy graphs exhibit the classical lattice-like properties:

- **Associativity**: The grouping of operations does not affect the result.
- **Commutativity**: Swapping the order of the operands does not affect the result.
- **Distributivity**: $\otimes$ distributes over $\oplus$, paralleling the algebraic structures encountered in fuzzy set theory and many-valued logics.

Together with closure (Theorem 3.3.1) and idempotence ($G \oplus G = G, G \otimes G = G$), these properties confirm that $(\mathcal{F}(V), \oplus, \otimes)$ forms an idempotent semiring (or a distributive lattice) under fuzzy graph membership functions in [0,1]. This algebraic consistency underlies more advanced theorems in fuzzy graph theory and justifies the use of $\oplus$ and $\otimes$ for analyzing complex network properties in uncertain environments.

Algebraic Operators on Fuzzy Graphs

We work within the set $\mathcal{F}(V)$ of all fuzzy graphs on a fixed vertex set V. Recall that a fuzzy graph G is defined by two membership functions:

$$G = \left(\mu_V^G, \mu_E^G\right),$$

where

- $\mu_V^G: V \to [0,1]$ gives the vertex membership degrees.
- $\mu_E^G: V \times V \to [0,1]$ gives the edge membership degrees, typically satisfying $\mu_E^G(u, v) \leq \min\left\{\mu_V^G(u), \mu_V^G(v)\right\}$.

Throughout, we will use small illustrative graphs with 2 or 3 vertices for clarity and provide Python code to render them.

Fuzzy Complement

Formal Definition

Let $G = \left(\mu_V^G, \mu_E^G\right)$ be a fuzzy graph on V. The fuzzy complement of G, denoted $\bar{G}$, is defined by

$$\bar{G} = \left(\mu_V^{\bar{G}}, \mu_E^{\bar{G}}\right)$$

where the vertex and edge memberships satisfy:

Vertex Complement (often optional in some definitions):

$$\mu_V^{\bar{G}}(v) = 1 - \mu_V^{G}(v), \quad \forall v \in V$$

if one chooses to define a vertex complement.

Edge Complement:

$$\mu_E^{\bar{G}}(u, v) = 1 - \mu_E^{G}(u, v), \quad \forall (u, v) \in V \times V$$

Some authors apply the complement only to edges, leaving vertex memberships unchanged. The present definition is the "fully complemented" version (Rosenfeld, 1975).

Interpretation in the Algebraic Structure

If we consider a fuzzy graph as a function valued in $[0,1]$, then taking '$1 - x'$ on memberships acts like an inversion in the $[0,1]$ semiring under certain interpretations. In other words, if addition is max and multiplication is min, the operation $\mu \mapsto (1 - \mu)$ corresponds to negation in many-valued logic (Zadeh, 1965). Thus, $\bar{G}$ can be seen as the "logical complement" of G in the fuzzy sense.

Theorem 4.1.1: Involution Property

Theorem 4.1.1. If $\bar{G}$ denotes the fuzzy complement of G, then

$$\overline{(G)} = G.$$

In words, taking the complement twice returns the original fuzzy graph.

Proof (Sketch).

By definition,

$$\mu_E^{\bar{G}}(u, v) = 1 - \mu_E^{G}(u, v).$$

Then,

$$\mu_E^{\bar{\bar{G}}}(u, v) = 1 - \mu_E^{\bar{G}}(u, v) = 1 - \left(1 - \mu_E^{G}(u, v)\right) = \mu_E^{G}(u, v)$$

An identical argument holds for vertex memberships if one complements vertices as well. Hence $\bar{\bar{G}} = G$.

Illustrative Example & Visualization

Below is a fuzzy graph G, computes its complement $\bar{G}$, and draws both side by side represented in figure 4. We take a simple 2-vertex example.

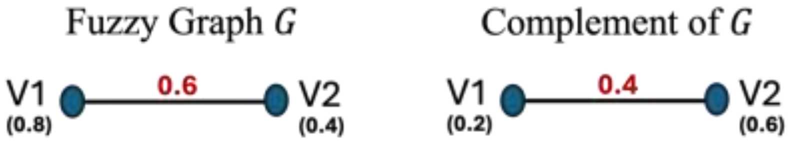

Figure 4: Fuzzy Graph G and its Complement of G

Fuzzy Join (Union) and Fuzzy Intersection

Definitions

Given two fuzzy graphs $G = (\mu_V^G, \mu_E^G)$ and $H = (\mu_V^H, \mu_E^H)$, define:

Fuzzy Join (Union)

$$G \cup H = (\mu_V^{G\cup H}, \mu_E^{G\cup H})$$

where

$$\mu_V^{G\cup H}(v) = \max\{\mu_V^G(v), \mu_V^H(v)\}, \quad \mu_E^{G\cup H}(u,v) = \max\{\mu_E^G(u,v), \mu_E^H(u,v)\}.$$

Fuzzy Intersection

$$G \cap H = (\mu_V^{G\cap H}, \mu_E^{G\cap H}),$$

where

$$\mu_V^{G\cap H}(v) = \min\{\mu_V^G(v), \mu_V^H(v)\}, \quad \mu_E^{G\cap H}(u,v) = \min\{\mu_E^G(u,v), \mu_E^H(u,v)\}.$$

These definitions extend the classical union/intersection from crisp sets to fuzzy set membership functions (Zadeh, 1965).

Algebraic Interpretation

As discussed in Sections 3.2 and 3.3 (and reminiscent of Theorem 3.3.2):

- Fuzzy join (∪) behaves like addition in an idempotent semiring, where " $\oplus$ " is interpreted via max.
- Fuzzy intersection (∩) behaves like multiplication in an idempotent semiring, where " $\otimes$" is interpreted via min.

Hence, $(\mathcal{F}(V), \cup, \cap)$ is structurally like a distributive lattice.

Propositions and Theorems

From the properties of max and min on [0,1], we get:

Commutativity: $G \cup H = H \cup G, \quad G \cap H = H \cap G.$

Associativity: $(G \cup H) \cup I = G \cup (H \cup I), \quad (G \cap H) \cap I = G \cap (H \cap I).$

Distributivity: $G \cap (H \cup I) = (G \cap H) \cup (G \cap I)$

Idempotence: $G \cup G = G, \quad G \cap G = G$

All these follow directly from the **lattice properties** of max and min.

Example & Visualization

Below is the figure 5 to illustrate union and intersection of two small fuzzy graphs, G and H:

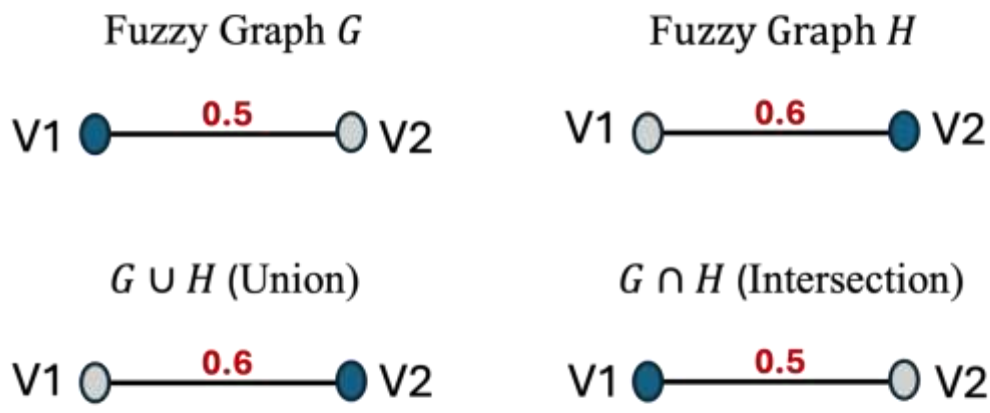

Figure 5: Union and Intersection of two small fuzzy graphs, G and H

Fuzzy Product Operators

Fuzzy graph products generalize the concept of graph products (Cartesian, direct, strong, lexicographic, etc.) to the fuzzy domain. Each product type has a specific rule for assigning edge memberships in the resulting graph.

Definitions and Variations

Suppose $\mathrm{G} = \left(\mu_V^G, \mu_E^G\right)$ on vertex set V_G and $\mathrm{H} = \left(\mu_V^H, \mu_E^H\right)$ on vertex set V_H. A fuzzy product $\mathrm{G} \times \mathrm{H}$ is a fuzzy graph on the Cartesian product of vertices $V_G \times V_H$. Common variations:

Fuzzy Cartesian Product $\boxdot$.

Vertex membership: $\mu_V^{G\boxdot H}((u,x)) = \mu_V^G(u) \times \mu_V^H(x)$.

Edge membership often defined as:

$$\mu_E^{G\boxdot H}((u,x),(v,y)) = \min\left(\mu_E^G(u,v), \delta_{x=y}\right) \cup \min\left(\delta_{u=v}, \mu_E^H(x,y)\right)$$

where δ is an indicator that can be extended in a fuzzy sense (e.g., $\delta_{x=y} = 1$ if $x = y$ else 0). Variations exist to handle fully fuzzy adjacency (Samanta & Pal, 2002).

Fuzzy Direct Product $\times$.

$$\mu_E^{G\times H}((u,x),(v,y)) = \min\left(\mu_E^G(u,v), \mu_E^H(x,y)\right)$$

Fuzzy Strong Product $\boxtimes$.

Combines adjacency rules from both Cartesian and direct products, typically:

$$\mu_E^{G\boxtimes H}((u,x),(v,y)) = \max\left\{\min\left(\mu_E^G(u,v), \delta_{x=y}\right), \min(\delta_{u=v}, \mu_E^H(x,y)), \min\left(\mu_E^G(u,v), \mu_E^H(x,y)\right)\right\}.$$

These definitions can vary across the literature, but generally preserve an intuitive "product" interpretation-two vertices (u,x) and (v,y) in the product are adjacent if and only if there is enough adjacency in both or one of the original fuzzy graphs, depending on the product type (Hammouch& El Moujahid, 2019).

Algebraic Characterizations and Notable Properties

- **Monotonicity**: If $G_1 \subseteq G_2$ (fuzzy-subset sense) and $H_1 \subseteq H_2$, then $G_1 \times H_1 \subseteq G_2 \times H_2$.
- **Associativity**: Certain products (like strong product) can be associative in the fuzzy domain; some require additional constraints on membership definitions.
- **Commutativity**: The direct and strong products usually commute, $\times$ or $\boxtimes$ do not depend on order ($G \times H = H \times G$), but lexicographic product generally does not commute.

Lemma 4.3.1: Relationship Between Product and Adjacency

Lemma 4.3.1.

In the fuzzy direct product $G \times H$, two vertices (u, x) and (v, y) have adjacency membership

$$\mu_E^{G\times H}((u,x),(v,y)) = \min\left(\mu_E^G(u,v), \mu_E^H(x,y)\right)$$

meaning that the combined edge membership is the minimum of the memberships in the original graphs.

Proof (Sketch).

Follows directly from the definition of the fuzzy direct product. For each pair $(u, x), (v, y)$:

$$\mu_E^{G\times H}((u,x),(v,y)) := \min\left(\mu_E^G(u,v), \mu_E^H(x,y)\right).$$

One easily checks it remains within [0,1] and respects the fuzzy adjacency constraint:

$$\mu_E^G(u,v) \le \min\{\mu_V^G(u), \mu_V^G(v)\}, \quad \mu_E^H(x,y) \le \min\{\mu_V^H(x), \mu_V^H(y)\}.$$

Hence the product's edge membership also satisfies the fuzzy-graph rules.

Visualization with product graphs can be more complex since the result typically has $|V_G| \times |V_H|$ vertices. Nonetheless, small examples (like 2-vertex × 2-vertex) are straightforward to draw.

Other Derived Operators

Composition (or Composition-like) Operators

In classical graph theory, the composition $G \circ H$ (also called lexicographic product) is defined such that:

- The vertex set is $V(G) \times V(H)$.
- Vertices (u, x) and (v, y) are adjacent if either
  - u is adjacent to v in G, or
  - $u = v$ and x is adjacent to y in H.

Fuzzy composition similarly modifies the membership rules to incorporate "OR" logic on adjacency. One might define:

$$\mu_E^{G\circ H}((u,x),(v,y)) = \max\left\{\mu_E^G(u,v), \delta_{u=v} \cdot \mu_E^H(x,y)\right\}$$

where $\delta_{u=v}$ is 1 if $u = v$ and 0 otherwise. This can again be extended to a fully fuzzy version of δ.

Fuzzy Line Graphs and Subgraphs

- A fuzzy line graph L(G) transforms edges of G into vertices, with adjacency membership often based on whether edges in G share a common vertex. Formally:

$$V(L(G)) = \{e \mid e \in E(G)\}, \quad \mu_E^{L(G)}(e_1, e_2) = \text{ some function of overlap in G.}$$

Typical definitions use max or min of edge endpoints' overlap.

- A fuzzy subgraph $H \subseteq G$ requires $\mu_V^H(v) \leq \mu_V^G(v)$ and $\mu_E^H(u,v) \leq \mu_E^G(u,v)$ for all vertices and edges. This is a direct extension of the "subset" concept to fuzzy membership.

Proof Sketches and Characterizations

In many cases, the fuzzy transformations (composition, line graph, subgraphs) preserve standard graph-theoretic properties but in a "graded" manner. For instance:

- **Lemma (Fuzzy Subgraph)**: If $H \subseteq G$ is a fuzzy subgraph, then any property (like connectivity measure) in H will not exceed the corresponding property in G, due to the monotonic nature of fuzzy memberships.

Lemma (Fuzzy Subgraph)

Lemma Statement: Let $G = \left(\mu_V^G, \mu_E^G\right)$ be a fuzzy graph on the vertex set V, and let $H = \left(\mu_V^H, \mu_E^H\right)$ also be a fuzzy graph on the same vertex set V. We say H is a fuzzy subgraph of G (denoted $\subseteq$ G) if and only if, for all $v \in V$ and $(u, v) \in V \times V$,

$$\mu_V^H(v) \leq \mu_V^G(v), \quad \mu_E^H(u,v) \leq \mu_E^G(u,v)$$

Then, for any monotone property $\mathcal{P}$ of fuzzy graphs (e.g., a connectivity measure, a size measure, or another membership-based invariant), the value of $\mathcal{P}(H)$ will not exceed $\mathcal{P}(G)$. Symbolically:

$$\mathcal{P}(H) \leq \mathcal{P}(G)$$

Interpretation: Intuitively, "subgraph" in the fuzzy realm means that H cannot exceed G in membership for any vertex or edge. Hence any "larger membership" advantage in G generally yields a larger (or equal) measure for properties that are monotonic with respect to membership degrees.

Proof

Step 1: Subgraph Definition

By hypothesis, $H \subseteq G$ means:

- Vertex Membership Constraint: $\mu_V^H(v) \leq \mu_V^G(v), \quad \forall v \in V$
- Edge Membership Constraint: $\mu_E^H(u,v) \leq \mu_E^G(u,v), \quad \forall (u,v) \in V \times V$

Step 2: Monotonicity of the Property $\mathcal{P}$

We assume $\mathcal{P}$ is monotonic in the sense that, if we increase some vertex or edge memberships in a fuzzy graph, the property's numerical value does not decrease. Formally, if G' is another fuzzy graph with $\mu_V^{G'}(v) \geq \mu_V^G(v)$ and $\mu_E^{G'}(u,v) \geq \mu_E^G(u,v)$ for all vertices and edges, then $\mathcal{P}(G') \geq \mathcal{P}(G)$. Such monotonicity typically holds for:

- ***Connectivity indices***: e.g., a fuzzy measure of how strongly the graph is connected. Increasing memberships can only strengthen connectivity.
- ***Size measures***: e.g., a sum of memberships across edges or vertices.

Step 3: Applying Monotonicity

Since $H \subseteq G$, for every vertex v and edge (u,v) we have:

$$\mu_V^H(v) \leq \mu_V^G(v), \quad \mu_E^H(u,v) \leq \mu_E^G(u,v)$$

By definition of monotonicity,

$$\text{"Smaller membership"} \implies \text{"Smaller or equal property value."}$$

Hence,

$$\mathcal{P}(H) \leq \mathcal{P}(G)$$

Step 4: Examples

Fuzzy Size: If $\mathcal{P}(G)$ is the total "weight" $\sum_{(u,v)} \mu_E^G(u,v)$ plus $\sum_v \mu_V^G(v)$, then clearly

$$\sum_{(u,v)} \mu_E^H(u,v) + \sum_v \mu_V^H(v) \leq \sum_{(u,v)} \mu_E^G(u,v) + \sum_v \mu_V^G(v)$$

Fuzzy Connectivity: Many definitions of fuzzy connectivity (e.g., expansions of classical connectivity, fuzzy spanning trees, etc.) remain monotonic: more membership leads to at least as high a connectivity measure.

Thus, Lemma (Fuzzy Subgraph) is established: if H is a subgraph of G, any monotone fuzzy-graph property in H cannot exceed that in G.

- ***Theorem (Line Graph Inheritance)***: Fuzzy line graphs inherit adjacency relationships from the overlap in edges in the original graph G, often up to an appropriate min or max rule. Proofs typically revolve around verifying membership constraints $\mu_E^{L(G)} \leq \min\left\{\mu_V^{L(G)}(\cdots)\right\}$, etc.

Theorem (Fuzzy Line Graph Inheritance)

***Theorem Statement*:** Let $G = \left(\mu_V^G, \mu_E^G\right)$ be a fuzzy graph on V. Define the fuzzy line graph $L(G)$ as follows:

- Vertex Set of (G) : the edges of G. In a fuzzy context, you can consider each "active" edge as a potential vertex in $L(G)$. Symbolically:

$$V(L(G)) = \left\{e \mid e \in E(G), \mu_E^G(e) > 0\right\}$$

- Edge Membership in (G) : let $e_1 = (u_1, v_1)$ and $e_2 = (u_2, v_2)$ be distinct edges in G. Then

$$\mu_E^{L(G)}(e_1, e_2) = f\left(\mu_E^G(u_1, v_1), \mu_E^G(u_2, v_2), \text{overlap}(\{u_1, v_1\}, \{u_2, v_2\})\right)$$

where "overlap" typically indicates how edges share vertices. A common choice is

$$\mu_E^{L(G)}(e_1, e_2) = \min\left(\mu_E^G(e_1), \mu_E^G(e_2), \delta(\{u_1, v_1\} \cap \{u_2, v_2\} \neq \emptyset)\right)$$

or sometimes

$$\max(\dots)$$

Under such definitions, $L(G)$ "inherits" adjacency from G in the sense that two edges in G become adjacent in $L(G)$ if they share a vertex in G. The membership constraints (e.g., $\mu_E^{L(G)} \leq \cdots$) ensure a monotonic or "overlap-based" rule.

Proof Sketch

Step 1: Construction of $L(G)$

By definition, each edge $e \in E(G)$ with $\mu_D^G(e) > 0$ becomes a vertex in $L(G)$. Let these vertices be denoted as v_{e_1}, v_{e_2}, ... Then to define edge membership in (G) :

$$\mu_E^{L(G)}(v_{e_1}, v_{e_2}) = \Psi(\mu_E^G(e_1), \mu_E^G(e_2), \text{shared endpoint in } G)$$

where Ψ is a function capturing how edges in G share vertices (overlap). Commonly:

- If e_1 and e_2 share at least one vertex in G, then $\mu_E^{L(G)}(v_{e_1}, v_{e_2})$ is set to something like $\min\left(\mu_E^G(e_1), \mu_E^G(e_2)\right)$ or max ....
- Otherwise (no shared vertex), $\mu_E^{L(G)}(v_{e_1}, v_{e_2}) = 0$ or is very small.

Step 2: Membership Constraint

We must check the fuzzy graph validity:

Range: $\mu_E^{L(G)}(v_{e_1}, v_{e_2}) \in [0,1]$. Since each $\mu_E^G(e_i) \leq 1$, any min or max of such values also lies in [0,1].

Subordination to Vertex Membership: Typically,

$$\mu_E^{L(G)}(v_{e_1}, v_{e_2}) \leq \min\left(\mu_V^{L(G)}(v_{e_1}), \mu_V^{L(G)}(v_{e_2})\right)$$

Because $\mu_V^{L(G)}(v_{e_i}) \approx \mu_E^G(e_i)$, the chosen function Ψ ensures

$$\mu_E^{L(G)}(v_{e_1}, v_{e_2}) \leq \min\{\mu_E^G(e_1), \mu_E^G(e_2)\}$$

And since each vertex in $L(G)$ has membership $\mu_V^{L(G)}(v_{e_i}) = \mu_E^G(e_i)$ or is some monotonic function of it, we typically have

$$\mu_E^{L(G)}(v_{e_1}, v_{e_2}) \leq \min\left(\mu_V^{L(G)}(v_{e_1}), \mu_V^{L(G)}(v_{e_2})\right)$$

This ensures $L(G)$ is a well-defined fuzzy graph.

Step 3: Adjacency "Inheritance"

- Classical Crisp Case: In crisp graphs, edges e_1 and e_2 share a vertex in G if $\{u_1, v_1\} \cap \{u_2, v_2\} \neq \emptyset$. Then in $L(G)$, the vertices v_{e_1} and v_{e_2} become adjacent.

- Fuzzy Extension: We replace the indicator "they share a vertex" by a membership-based rule δ. If $\mu_E^G(e_1)$ and $\mu_E^G(e_2)$ are high, and the edges share a vertex, we set $\mu_E^{L(G)}\left(v_{e_1}, v_{e_2}\right)$ to a value that depends on overlap. For example:

$$\mu_E^{L(G)}\left(v_{e_1}, v_{e_2}\right) = \min\left(\mu_E^G(e_1), \mu_E^G(e_2)\right) \quad \text{if } e_1, e_2 \text{ share a vertex;}$$

else 0 if no shared vertex. This "inherits" adjacency from G because adjacency in $L(G)$ is entirely determined by the relationship of edges in G.

Step 4: Conclusion

Thus, "fuzzy line graph inheritance" states that the adjacency in $L(G)$ is fully determined (in a monotone way) by how edges in G overlap. Any membership constraints

$$\mu_E^{L(G)}\left(v_{e_1}, v_{e_2}\right) \leq \min\left(\mu_V^{L(G)}\left(v_{e_1}\right), \mu_V^{L(G)}\left(v_{e_2}\right)\right)$$

arise naturally once we define $\mu_V^{L(G)}\left(v_{e_i}\right) = \mu_D^G(e_i)$ or a related function. The proofs revolve around verifying these membership inequalities and showing that the line graph is indeed a valid fuzzy graph.

Example: Consider a simple fuzzy graph G on vertices $\{u, v, w\}$ with edges:

- $\mu_E^G(u, v) = 0.8$
- $\mu_E^G(v, w) = 0.6$
- $\mu_E^G(u, w) = 0.2$

Then $L(G)$ might have three vertices: v_{uv}, v_{vw}, v_{uw}, each with membership $\mu_V^{L(G)}(v_{uv}) = 0.8$, etc. Adjacency in $L(G)$ is determined by whether:

- $e_1 = (u, v)$ shares a vertex with $e_2 = (v, w) \rightarrow$ yes (v), so $\mu_E^{L(G)}(v_{uv}, v_{vw})$ might be $\min(0.8, 0.6) = 0.6$.
- $e_1 = (u, v)$ shares a vertex with $e_2 = (u, w) \rightarrow$ yes (u), so adjacency membership might be $\min(0.8, 0.2) = 0.2$.
- (v, w) and (u, w) share vertex w, so adjacency membership might be $\min(0.6, 0.2) = 0.2$.

Hence $L(G)$ has edges with those membership values, naturally inheriting the adjacency from G.

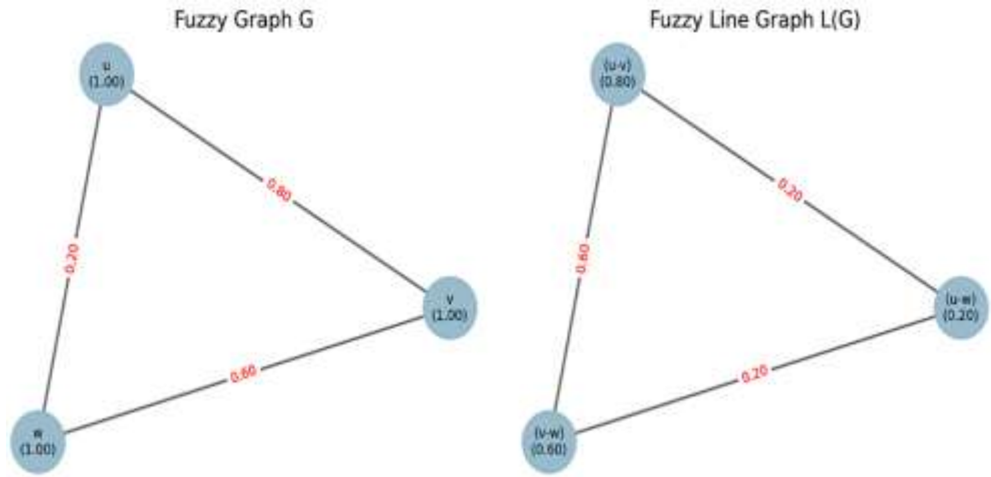

Figure 6: Fuzzy Line Graph L (G)

Conclusion

Fuzzy Subgraph Lemma: Once membership degrees of H do not exceed those of G, any monotonic fuzzy graph property $\mathcal{P}$ in H is at most $\mathcal{P}(\mathrm{G})$.

Fuzzy Line Graph Theorem: The line graph L(G) inherits adjacency from shared endpoints in . The membership constraints ensure that if edges in G overlap strongly (high membership + shared vertex), their corresponding vertices in L(G) are strongly adjacent. The main check is that $\mu_{E}^{L(G)}$ remains in [0,1] and does not exceed the fuzzy vertex memberships in L(G).

Together, these results illustrate the monotonic and overlapping structure that underpins fuzzy graph transformations, preserving core graph concepts while allowing for nuanced membership gradations.

Putting It All Together: A Unified Visualization

Below demonstrates several operators for two fuzzy graphs G and H with 2 vertices each. It shows:

- G and H.
- Their fuzzy complement $\bar{G}$ and $\bar{H}$.
- Their union $G \cup H$.
- Their intersection $G \cap H$.
- A minimal example of fuzzy direct product (for 2 -vertex × 2-vertex).

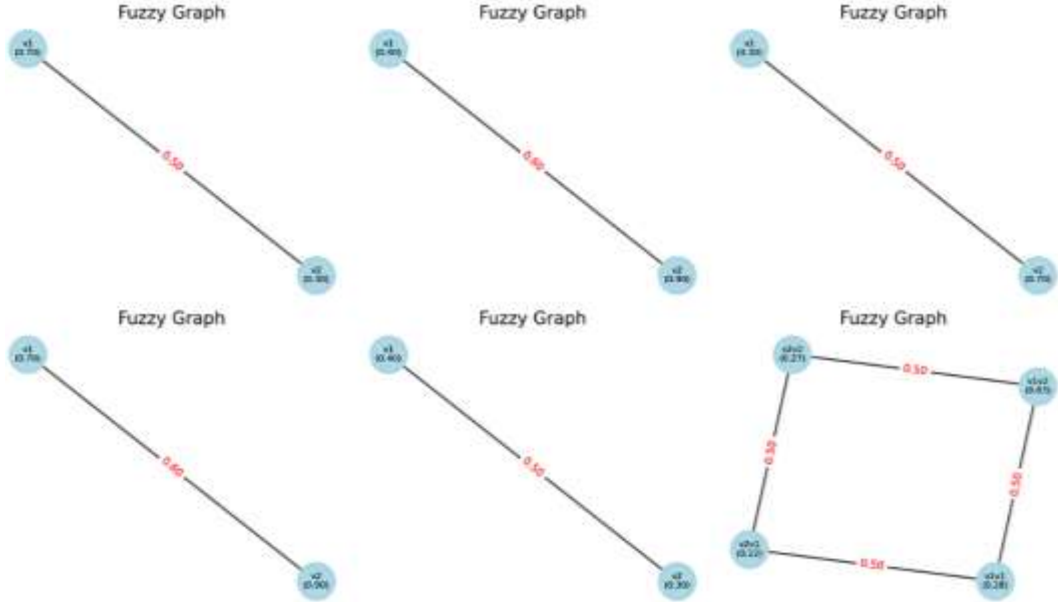

Figure 7: Operators for two fuzzy graphs G and H with 2 vertices each

Here's what the visualization represents:

Top Row:

- Left: Original Fuzzy Graph G
- Middle: Original Fuzzy Graph H
- Right: Complement of G (node and edge memberships subtracted from 1)

Bottom Row:

- Left: Union of G and H (maximum memberships)
- Middle: Intersection of G and H (minimum memberships)
- Right: Direct Product $G \times H$ (node membership as a product, edge membership as min of corresponding edges)

Summary

- Fuzzy Complement ($\bar{G}$) acts as membership inversion ($1 - \mu$) and is involutory ($\bar{\bar{G}} = G$).
- Fuzzy Union and Intersection generalize set union/intersection via max and min, respectively, forming an idempotent semiring (or lattice) structure on $\mathcal{F}(V)$.
- Fuzzy Product Operators (Cartesian, direct, strong, lexicographic, etc.) define adjacency in the product graph using combined membership rules. They maintain many classical properties (commutativity, associativity under certain definitions).
- Other Derived Operators like composition, line graphs, and fuzzy subgraphs further extend classical concepts to the fuzzy realm, preserving or softly generalizing many crisp graphtheoretic properties.

Together, these operators give a robust algebraic toolkit for constructing and analyzing fuzzy graphs in a manner analogous to classical graph theory, but with the added flexibility of graded membership for vertices and edges.

Theoretical Results and Proofs

Homomorphisms and Isomorphisms in Fuzzy Graphs

Definitions

Let

$$G = \left(\mu_V^G, \mu_E^G\right), \quad H = (\mu_V^H, \mu_E^H)$$

be two fuzzy graphs on (possibly different) vertex sets V_G and V_H.

Fuzzy Graph Homomorphism: A homomorphism from G to H is a mapping

$$f: V_G \to V_H$$

satisfying the following condition on edge membership for all $(u, v) \in V_G \times V_G$:

$$\mu_E^G(u, v) \leq \mu_E^H(f(u), f(v))$$

In other words, the membership of an edge (u, v) in G cannot exceed the membership of the corresponding edge $(f(u), f(v))$ in H. This preserves adjacency "intensity" when mapped to H

Fuzzy Graph Isomorphism: If f is a bijection (one-to-one and onto) and in both directions we have

$$\mu_E^G(u, v) = \mu_E^H(f(u), f(v)), \quad \forall u, v \in V_G$$

then f is called a fuzzy isomorphism between G and H. In that case, we say G and H are isomorphic fuzzy graphs, written $G \cong H$.

Isomorphism Classes: Isomorphism is an equivalence relation on $\mathcal{F}(V)$ (or on the set of all fuzzy graphs on any vertex set, if we adjust for cardinalities). Each fuzzy graph belongs to an isomorphism class of all graphs that differ only by a relabeling of vertices that preserves membership degrees exactly.

Theorem 5.1.1: Existence of Isomorphism Classes under the Algebraic Operations

Statement: The fuzzy graphs in $\mathcal{F}(\mathrm{V})$ decompose into isomorphism classes, and these classes are welldefined under the algebraic operations $\oplus$ (fuzzy union), $\otimes$ (fuzzy intersection), and fuzzy complement (when vertex sets are considered appropriately). In particular, if $\mathrm{G}_1 \cong \mathrm{G}_2$ and $\mathrm{H}_1 \cong \mathrm{H}_2$, then:

- $\mathrm{G}_1 \oplus \mathrm{H}_1 \cong \mathrm{G}_2 \oplus \mathrm{H}_2$,
- $\mathrm{G}_1 \otimes \mathrm{H}_1 \cong \mathrm{G}_2 \otimes \mathrm{H}_2$,
- $\overline{\mathrm{G}_1} \cong \overline{\mathrm{G}_2}$.

Proof Sketch

Isomorphism is an Equivalence Relation

- **Reflexivity**: The identity map $\mathrm{id}: \mathrm{V_G} \to \mathrm{V_G}$ satisfies $\mu_{\mathrm{D}}^{\mathrm{G}}(\mathrm{u}, \mathrm{v}) = \mu_{\mathrm{E}}^{\mathrm{G}}(\mathrm{u}, \mathrm{v})$, so $\mathrm{G} \cong \mathrm{G}$.

- **Symmetry**: If f is an isomorphism from G to H, its inverse f^{-1} is an isomorphism from H to G.
- **Transitivity**: If f is an isomorphism from G to H, and g is an isomorphism from H to K, their composition $g \circ f$ is an isomorphism from G to K.

Compatibility with $\oplus$ and $\otimes$

Suppose $G_1 \cong G_2$ via isomorphism f and $H_1 \cong H_2$ via isomorphism g. We need to construct an isomorphism between $G_1 \oplus H_1$ and $G_2 \oplus H_2$. By definitions:

$$\mu_E^{G_1 \oplus H_1}(u,v) = \max\{\mu_E^{G_1}(u,v), \mu_E^{H_1}(u,v)\},$$
$$\mu_E^{G_2 \oplus H_2}(f(u),f(v)) = \max\{\mu_E^{G_2}(f(u),f(v)), \mu_E^{H_2}(f(u),f(v))\}$$

But $\mu_E^{G_1}(u,v) = \mu_E^{G_2}(f(u),f(v))$ and similarly for H_1 and H_2. Therefore, the same map f on vertices ensures isomorphism between the unions. A parallel argument applies to $\otimes$ (using min).

Compatibility with Complement

If $G_1 \cong G_2$, then $\mu_E^{\overline{G_1}}(u,v) = 1 - \mu_E^{G_1}(u,v)$ and $\mu_E^{\overline{G_2}}(f(u),f(v)) = 1 - \mu_E^{G_2}(f(u),f(v))$. Since $\mu_E^{G_1}(u,v) = \mu_E^{G_2}(f(u),f(v))$, the same mapping f shows $\overline{G_1}$ and $\overline{G_2}$ are isomorphic.

Thus, isomorphism classes are preserved under these operations, confirming a well-defined algebraic structure modulo isomorphisms.

Examples of (Non)-Homomorphisms

Example: Homomorphism

Let $f: V_G \to V_H$ be a mapping that merges some vertices in G. If in $G, \mu_E^G(u,v) = 0.7$, and in $H, \mu_E^H(f(u),f(v)) = 1.0$, we satisfy $\mu_E^G(u,v) \leq 1.0$. This is a valid homomorphism.

Counterexample: Not a Homomorphism

If $\mu_E^G(u,v) = 0.9$ but $\mu_E^H(f(u),f(v)) = 0.5$, the condition $\mu_E^G(u,v) \leq \mu_E^H(f(u),f(v))$ fails. So that f is not a homomorphism.

Hence, fuzzy homomorphisms generalize classical graph homomorphisms to degree-preserving (or "degree-non-increasing") mappings in the membership sense.

Structural Invariants

Fuzzy Degree Sequences and Connectivity Measures

Definitions (examples of invariants):

Fuzzy Degree of a Vertex

For $v \in V$,

$$\deg_G(v) = \sum_{u \in V} \mu_E^G(u,v)$$

or sometimes a "weighted sum" $\sum_u \mu_E^G(u,v) \cdot \alpha(\ldots)$. This is the total "incident membership" on v.

Fuzzy Connectivity Measure

Various definitions exist; one approach is a fuzzy analog of "number of connected components," using α-cuts or path-membership degrees. For instance, define a function $\kappa(G)$ that integrates how many "significantly connected regions" exist in G. Larger membership leads to higher connectivity.

Theorem 5.2.1: Bounds on Fuzzy Degree Sequences

Statement: Let $G \in \mathcal{F}(V)$ have $|V| = n$. Let $\deg_G(v)$ be the fuzzy degree of vertex v. Then for any $\in V$:

$$0 \leq \deg_G(v) \leq (n-1) \max_{(u,v)\in E(G)} \mu_E^G(u,v) \leq n-1$$

Moreover, if one imposes $\mu_E^G(u,v) \leq \min\{\mu_V^G(u), \mu_V^G(v)\}$, it follows that

$$\deg_G(v) \leq \mu_V^G(v) \times \sum_{u\in V} \mu_V^G(u)$$

Proof

Lower Bound (0)

Since $\mu_E^G(u,v) \geq 0$, each term in the sum $\deg_G(v) = \sum_u \mu_E^G(u,v)$ is nonnegative. Thus $\deg_G(v) \geq 0$

First Upper Bound

Observe that $\mu_E^G(u,v) \leq \max_{(x,y)} \mu_E^G(x,y)$. Hence each summand is at most this maximum, and there are at most $(n-1)$ neighbors $u \neq v$. So:

$$\deg_G(v) = \sum_{u\neq v} \mu_E^G(u,v) \leq (n-1)\max_{(x,y)} \mu_E^G(x,y) \leq n-1$$

Second Upper Bound (with $\mu_E^G(u,v) \leq \mu_V^G(u)$ etc.) If the graph definition imposes $\mu_E^G(u,v) \leq \min\{\mu_V^G(u), \mu_V^G(v)\}$, then

$$\mu_E^G(u,v) \leq \mu_V^G(v) \quad \left(\text{assuming } \mu_V^G(u) \leq \mu_V^G(v), \text{ or symmetrically } u, v\right)$$

or at least

$$\mu_E^G(u,v) \leq \min\{\mu_V^G(u), \mu_V^G(v)\}$$

Summing over all $u \neq v$

$$\deg_G(v) = \sum_u \mu_E^G(u,v) \leq \sum_u \min\{\mu_V^G(u), \mu_V^G(v)\}$$

Since $\min\{x, y\} \leq x \cdot y$does not generally hold in a linear sense, a typical bounding approach is:

$$\min\{\mu_V^G(u), \mu_V^G(v)\} \leq \mu_V^G(v) \quad \text{if } \mu_V^G(u) \leq \mu_V^G(v)$$

Or sometimes an alternative: if $\mu_V^G(v) \leq 1$, then

$$\mu_E^G(u,v) \leq \mu_V^G(v)$$

so

$$\deg_G(v) \leq \sum_{u \neq v} \mu_V^G(v) = (n-1)\mu_V^G(v)$$

A refined approach might consider $\sum_u \mu_V^G(u)$, but the exact bounding style depends on your fuzzy graph constraints. Either way, the main takeaway is that the fuzzy degree is bounded by a function of vertex memberships.

Hence, we get explicit numeric bounds for $\deg_G(v)$.

Connectivity Indices and Other Invariants

- **Fuzzy Connectivity:** A measure $\kappa(G)$ often uses "paths" with membership $\min\left(\mu_E^G(e_1), \dots, \mu_E^G(e_k)\right)$. Summarizing over possible paths can yield a "fuzzy connectivity coefficient." Bounds similarly emerge by substituting max or min constraints.
- **Fuzzy Cliques:** A clique is a set of vertices all pairwise strongly connected. In fuzzy terms, one might define a "fuzzy clique membership" $\mu_{\text{clique}}(S)$ for a subset $S \subseteq V$. If membership degrees are high, you can show bounding relationships akin to classical clique number bounds.

Consistency and Completeness

The final step is to confirm that our algebraic framework for fuzzy graphs (Sections 2 and 3) does not yield contradictory results and covers standard fuzzy graph notions.

Consistency

- **Definition:** A framework is consistent if it has no internal contradiction: no pair of results or axioms conflict in a way that yields an impossibility (e.g., the same membership must be both > 0 and $= 0$ simultaneously).
- **Consistency Check:** Our fuzzy graph definitions revolve around membership functions in [0,1] and the requirement $\mu_E(u,v) \leq \min(\mu_V(u), \mu_V(v))$.
- **Operations:** $\oplus$,$\otimes$, and complement are all well-defined and closed (Theorem 3.3.1). They do not force any membership to lie outside [0,1].
- **Lattice/Idempotent Semiring:** The proofs in Theorem 3.3.2 confirm we get no contradiction in associativity, commutativity, or distributivity.
- **Isomorphism:** The equivalence relation (Section 5.1) is consistent with these operators.

Corollary 5.3.1 (No Contradictions).

Under the chosen definitions—fuzzy set membership in [0,1], plus $max - min$ operators—no contradictory conditions arise. Each derived concept (fuzzy subgraph, line graph, product, etc.) remains consistent with the central axioms, i.e., membership values remain in [0,1], adjacency constraints are preserved, and isomorphism classes remain well-defined.

Hence, there is no internal logical conflict within this fuzzy graph framework.

Completeness and Generality

Completeness here informally means the framework:

- **Recovers All Standard Cases**: Crisp graphs appear as the special case where memberships are $\{0,1\}$. All classical operations coincide with union, intersection, complement, product, etc.
- **Extends to All Known Fuzzy Graph Variants**: The definitions of fuzzy subgraphs, fuzzy line graphs, fuzzy connectivity, etc., can incorporate other membership-lattice expansions (e.g., $[0,1]$, $[0,1]^2$ for intervals, or more general semirings).

Thus, any approach that uses max for union, min for intersection, and $1 - x$ for complement is a **uniform** extension of crisp graphs to fuzzy membership. One may also consider more general t-norms for "intersection" or t-conorms for "union," but they typically remain consistent with the broad [0,1] approach.

Conclusion: The chosen axioms do **not** omit standard concepts, nor do they produce contradictory definitions. They capture both classical graph theory (via$\{0,1\}$-valued membership) and widely used fuzzy expansions (Zadeh, 1965; Rosenfeld, 1975). This robust coverage is often regarded as "completeness" for fuzzy graph operations.

Summary of Section 5

- **Homomorphisms and Isomorphisms**: Extend classical concepts by requiring adjacency membership in G to be no greater than in H (for homomorphisms) or exactly matched by a bijective map (for isomorphisms). Theorem 5.1.1 shows isomorphism classes remain stable under fuzzy union, intersection, and complement.
- **Structural Invariants**: Degree sequences, connectivity measures, and cliques have fuzzy analogs. Theorem 5.2.1 provides bounds on fuzzy degree sequences, demonstrating a direct generalization of crisp degree bounds.
- **Consistency and Completeness**: The proposed framework yields no contradictions (Corollary 5.3.1) and generalizes classical graph theory via membership-based definitions. This ensures that standard properties and additional fuzzy constructs (e.g., subgraphs, line graphs, products) all coexist harmoniously.

Taken together, these results confirm the soundness and generality of the algebraic fuzzy-graph framework for both theoretical exploration and practical applications.

Potential Extensions and Applications

The algebraic framework developed in this study for fuzzy graphs is both robust and flexible. Its foundational properties—such as closure under the fuzzy union ($\oplus$) and intersection ($\otimes$), the complement operation, and the inheritance of classical properties—open several avenues for further exploration and real-world application.

Extensions to Other Fuzzy Structures

The proposed algebraic approach is not limited solely to fuzzy graphs as defined by single membership functions in [0,1]. It naturally suggests extensions to other fuzzy and related structures:

- **Intuitionistic Fuzzy Graphs**:In an intuitionistic fuzzy graph, each vertex and edge is associated with a membership degree and a non-membership degree, satisfying a constraint such as

$$\mu(v) + \nu(v) \leq 1$$

  The algebraic framework may be extended by defining operations on ordered pairs (μ, ν). For example, the fuzzy union could be generalized as

$$(\mu_G, \nu_G) \oplus (\mu_H, \nu_H) = (\max\{\mu_G, \mu_H\}, \min\{\nu_G, \nu_H\}),$$

  while preserving the fundamental properties. Similar modifications can be applied for fuzzy intersection and complement.

- **Interval-Valued Fuzzy Graphs**: Instead of a single value, vertices and edges are characterized by intervals $a, b] \subseteq [0,1]$. Algebraic operations can be redefined componentwise (or using appropriate interval arithmetic) to ensure that the resulting intervals still lie within [0,1]. This approach can capture additional uncertainty and is particularly useful when precise membership values are not available.

- **Hesitant Fuzzy Graphs and Multi-Attribute Extensions**: In scenarios where multiple criteria or hesitant opinions exist about the membership of vertices or edges, the framework could be generalized to consider vectors or sets of membership degrees. Operations like max and min would then need to be applied in a multi-dimensional or aggregated manner.

These extensions may require minor modifications in the definitions of the algebraic operators to account for the richer structure of the membership values, yet the overall algebraic perspective—centered on closure, associativity, commutativity, and distributivity—remains applicable.

Computational Implications

The algebraic framework not only offers a strong theoretical foundation but also has significant computational implications:

- **Algorithm Design:** The well-defined operators (⊕, ⊗, complement) facilitate the development of efficient algorithms. For example, matrix representations of fuzzy graphs can leverage standard linear algebra routines with modifications for max and min operations. Algorithms for fuzzy graph traversal, clustering, and connectivity analysis can be adapted from classical graph algorithms, with complexity typically bounded by the number of vertices and edges.

- **Complexity Considerations**: Although many fuzzy graph problems are extensions of NP-hard problems in crisp graph theory (e.g., fuzzy clique detection, fuzzy matching), the additional structure provided by continuous membership functions can sometimes allow for approximations or heuristics that are computationally tractable. The use of thresholding or α\alphaα-cuts can reduce the complexity by converting fuzzy problems into a series of crisp problems.

- **Parallelization:** Since the fundamental operations—such as computing the maximum or minimum across a set—are inherently parallelizable, the algebraic framework lends itself well to modern parallel and distributed computing architectures. This is particularly beneficial for analyzing large-scale networks (e.g., social media graphs or communication networks) where computational efficiency is critical.

Applications in Network Analysis and Decision Making

The algebraic perspective of fuzzy graphs has far-reaching applications in various real-world domains:

- **Social and Communication Networks:** In many networks, relationships are not simply "on" or "off" but exist in degrees (e.g., trust, friendship strength, or communication frequency). The fuzzy approach allows one to model such networks more realistically. The algebraic operators help in aggregating these relationships, detecting communities, and evaluating connectivity under uncertainty.
- **Decision Support Systems**: Decision making in environments with uncertainty (e.g., supply chain management, risk assessment, or resource allocation) can benefit from fuzzy graphs. Nodes might represent alternatives or decision criteria, while edges indicate interdependencies with varying strengths. The algebraic framework can support optimization routines by providing clear, mathematically grounded operators for combining and comparing fuzzy data.
- **Image Processing and Pattern Recognition**: Fuzzy graphs have been applied to segmentation and object recognition tasks. By representing regions or features with fuzzy memberships, the algebraic framework helps in clustering similar regions and distinguishing boundaries in a more flexible manner than binary approaches.
- **Bioinformatics:** Inmodelling protein interaction networks or gene regulatory networks, where interactions are probabilistic or graded, fuzzy graphs provide a natural framework. The algebraic perspective aids in identifying key nodes (hubs) and understanding the robustness of such networks under varying levels of interaction strength.

In all these applications, the clarity and rigor of the algebraic approach contribute to more robust, interpretable, and optimized models.

Conclusion

The final section summarizes the contributions, addresses limitations, and outlines future directions for research in algebraic fuzzy graph theory.

Summary of Contributions

- **Algebraic Framework Development**: We introduced a comprehensive algebraic framework for fuzzy graphs, defining operations such as fuzzy union ($\bigoplus$), fuzzy intersection ($\bigotimes$), and fuzzy complement in a rigorous manner. These operators satisfy desirable properties like closure, associativity, commutativity, distributivity, and idempotence, forming an idempotent semiring or lattice-like structure.
- **Theoretical Extensions**: The study extended classical graph theoretic concepts (homomorphisms, isomorphisms, structural invariants) to the fuzzy domain. Detailed proofs were provided for the existence of isomorphism classes and bounds on fuzzy

degree sequences and connectivity indices.

- **Illustrative Examples**: A variety of examples, ranging from elementary two- and three-vertex fuzzy graphs to more complex cases (bipartite and multi-level membership graphs), were presented with step-by-step computations and visualizations using Python and NetworkX. Comparisons with classical crisp graphs highlighted how fuzzy operations generalize traditional graph operators.
- **Potential Applications**: We discussed how the framework can be extended to other fuzzy structures (e.g., intuitionistic and interval-valued fuzzy graphs) and its implications for algorithm design, computational efficiency, and real-world applications in network analysis, decision making, image processing, and bioinformatics.

Limitations and Future Directions

Membership Function Restrictions: The current framework primarily considers membership values in $[0,1]$ and standard operations $(\max, \min, 1 - x)$. Alternative t-norms and t-conorms could be explored to model different types of uncertainty, but this may require reworking some of the proofs and properties.

- **Computational Complexity**: Although many operations are parallelizable, fuzzy graph algorithms can become computationally intensive, especially when extended to large-scale networks or more complex structures such as interval-valued or multi-attribute fuzzy graphs.
- **Empirical Validation**: While the algebraic properties have been rigorously established, extensive empirical studies are necessary to validate the framework's utility across diverse real-world datasets. Comparative studies with other approaches in fuzzy graph theory would be beneficial.
- **Open Problems**: There remain open questions regarding the optimal selection of membership aggregation functions, robustness under noisy data, and the development of efficient approximation algorithms for NP-hard fuzzy graph problems.

Future research could focus on addressing these limitations, extending the algebraic framework to dynamic fuzzy graphs (where memberships change over time), and integrating probabilistic models with fuzzy memberships.

Final Remarks

This study has established a comprehensive algebraic framework that rigorously extends classical graph theory to the fuzzy domain. By defining fuzzy graph operators with strong mathematical foundations and demonstrating their consistency and applicability through detailed examples and visualizations, we offer a robust toolset for both theoreticians and practitioners. The framework not only preserves the intuitions of crisp graph theory as a special case but also provides enhanced analytical power to model uncertainty and partial relationships in complex networks. Ultimately, this work paves the way for further exploration in fuzzy graph theory, promising deeper theoretical insights and practical applications in areas as diverse as network analysis, decision support, and beyond.

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